Topology and its Applications 49 (1993) 75-94
North-Holland

Characterizations of a metrizable space X such that every $A_n(X)$ is a k-space

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Received 11 July 1991
Revised 29 October 1991

Abstract

Yamada, K., Characterizations of a metrizable space X such that every $A_n(X)$ is a k-space, Topology and its Applications 49 (1993) 75-94.

Let X be a metrizable space, then we obtain the following results.

(1) The following are equivalent:
  (a) $A_n(X)$ is a k-space for each $n \in \mathbb{N}$,
  (b) $A_4(X)$ is a k-space,
  (c) the canonical mapping $i_n : (X \oplus -X \oplus \{0\})^n \to A_n(X)$ is quotient for each $n \in \mathbb{N}$,
  (d) i_4 is quotient,
  (e) either X is locally compact and the set X' of all nonisolated points in X is separable, or X' is compact.

(2) The following are equivalent:
  (a) $A_3(X)$ is a k-space,
  (b) i_3 is quotient,
  (c) X is locally compact or X' is compact.

(3) $A_2(X)$ is a k-space and i_2 is quotient.

These results contain the answers to the questions of T.H. Fay, E.T. Ordman and B.V.S. Thomas for free Abelian topological groups.

Keywords: Free Abelian topological groups, k-spaces, quotient mappings, metrizable spaces.

AMS(MOS) subj. Class.: 22A05, 54D50, 54H10.

Introduction

Let $F(X)$ and $A(X)$ be the free topological group and the free Abelian topological group over a Tychonoff space X, respectively [6, 11]. The topological structures of $F(X)$ and $A(X)$ are complicated. Indeed, only when a space X is discrete, $F(X)$ and $A(X)$ can be first countable or locally compact.

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One of the techniques of studying the topological structure of $F(X)$ $(A(X))$ is the canonical representation of the space $F(X)$ $(A(X))$ as the union of its closed subspaces $F_n(X)$ $(A_n(X))$ formed by reduced words whose lengths are less than or equal to n, where $n \in \mathbb{N}$ (= the set of natural numbers). And, the technique is often used. For example, Graev [6] showed that if a space X is compact, E is a closed subset of $F(X)$ if and only if $E \cap F_n(X)$ is a closed subset of $F_n(X)$ for each $n \in \mathbb{N}$. By the result, he showed an important result that $F(X)$ is Weil complete, i.e., it is complete relative to right and left uniform structures of this group. Graev's results were generalized in [12] and [10]: if a topological group G is a k_ω-space, then the group G is complete, and if X is a k_ω-space, then $F(X)$ is also a k_ω-space. Hence, if X is a k_ω-space, then $F(X)$ is Weil complete. In fact, Mack, Morris and Ordman [10] showed that if X is a k_ω-space, then $F_n(X)$ is a k_ω-space for each $n \in \mathbb{N}$ and also E is a closed subset of $F(X)$ if and only if $E \cap F_n(X)$ is a closed subset of $F_n(X)$ for each $n \in \mathbb{N}$. The class of k_ω-spaces includes not only compact spaces but also locally compact Lindelöf spaces, and is narrower than the class of σ-compact spaces.

$F(X)$ $(A(X))$ need not be a k-space even if X is a k-space. That is, Fay, Ordman and Thomas [5] showed that the free topological group of the space of rationals $\mathbb{Q}$ is not a k-space, in fact, they showed $F_3(\mathbb{Q})$ is not a k-space. On the other hand, Arhangel'skiĭ, Okunev and Pestov [3] showed the following characterizations that $F(X)$ $(A(X))$ is a k-space.

Theorem A. *If X is metrizable and X' is the set of all nonisolated points in X, then the following conditions are equivalent*:

(a) *$A(X)$ is a k-space,*

(b) *$A(X)$ is homeomorphic to a product of a k_ω-space with a discrete space,*

(c) *X is locally compact and X' is separable.*

Theorem B. *If X is metrizable, then the following conditions are equivalent*:

(a) *$F(X)$ is a k-space,*

(b) *$F(X)$ is a k_ω-space or discrete,*

(c) *X is locally compact separable or discrete.*

In the proof of these theorems, they used some concrete spaces such that the free (Abelian) topological groups over them are not k-spaces. For example, the Fréchet-Urysohn fan $V(\aleph_1)$ of cardinality $\aleph_1$, the hedgehog space $J(\aleph_0)$ of spininess $\aleph_0$ such that each spininess is a sequence which converges to the center point, and $Y = C \oplus \{x_\alpha : \alpha < \omega_1\}$, where C is a convergent sequence with its limit and $\{x_\alpha : \alpha < \omega_1\}$ is a discrete collection. It can be proved that neither $A_3(V(\aleph_1))$, $F_2(V(\aleph_1))$, $A(J(\aleph_0))$, $F(J(\aleph_0))$, nor $F_4(Y)$ is a k-space (cf. [3, 14]). Since, by most of their ways, it was shown that $F_n(X)$ $(A_n(X))$, as a closed subspace of $F(X)$ $(A(X))$, is not a k-space for some $n \in \mathbb{N}$, we naturally raise the following question: if $F_n(X)$ $(A_n(X))$ is a k-space for each $n \in \mathbb{N}$, then is $F(X)$ $(A(X))$ a k-space?

However, we obtained the negative answer for $A(X)$ [18]. Namely,

Theorem C. *$A_n(J(\kappa))$ is a k-space for each $n \in \mathbb{N}$ and each cardinality κ, but $A(J(\kappa))$ is not a k-space if $\kappa \geqslant \aleph_0$.*

Furthermore, we had a necessary condition of a metrizable space X that each $A_n(X)$ is a k-space [18].

Theorem D. *For a metrizable space X, if $A_n(X)$ is a k-space for each $n \in \mathbb{N}$, then the set X' of all nonisolated points in X is locally compact in X'.*

Although the subspaces $A_n(X)$, $n \in \mathbb{N}$, play an important role to investigate the topology of $A(X)$, by the above result, there is a gap between the k-property of $A(X)$ and the one of $A_n(X)$, $n \in \mathbb{N}$. In this paper, we shall give characterizations of a metrizable space X such that every $A_n(X)$ is a k-space. As a consequence, we have a stability theorem of the property of $A_n(X)$, $n \in \mathbb{N}$. Thus,

Theorem E. *For a metrizable space X, the following statements are equivalent:*

(a) *$A_4(X)$ is a k-space,*

(b) *every $A_n(X)$, $n \in \mathbb{N}$, is a k-space.*

Since the definition of free topological groups says nothing about the constructive form of open sets, i.e., an open neighborhood base of the unit element e, it is difficult to investigate the topological properties on $F(X)$ $(A(X))$ and $F_n(X)$ $(A_n(X))$. Tkačenko [16] gave an open neighborhood base Σ^* of e in $F(X)$. But the form of Σ^* is complicated. In Section 2, we introduce an alternative construction of an open neighborhood base $\mathcal{W}$ of the unit element 0 in $A(X)$. Furthermore, we give an open neighborhood base $\mathcal{W}_n$ of 0 in $A_{2n}(X)$ for each $n \in \mathbb{N}$, using $\mathcal{W}$. In Section 4, applying $\mathcal{W}_n$, we shall give characterizations that every $A_n(X)$ is a k-space. As a result, we have not only a stability theorem but also facts of $A_2(X)$ and $A_3(X)$ as follows.

Theorem F. (1) *$A_2(X)$ is a k-space for every metrizable space X, and there exists a metrizable space X such that $A_3(X)$ is not a k-space.*

(2) *There exists a metrizable space X such that $A_3(X)$ is a k-space but $A_4(X)$ is not a k-space.*

In the last section, Section 5, we shall answer to the questions of Fay, Ordman and Thomas [5] for $A(X)$.

1. Notations and preliminaries

All topological spaces are assumed to be Tychonoff. By $\mathbb{N}$ we denote the set of all natural numbers. Our terminology and notations follow [4], and we refer [7] for elementary properties of topological groups.

The *free topological group* $F(X)$ over a space X in the sense of Markov [11] is the free algebraic group over the set X equipped with the group topology $\mathcal{T}$ having the following properties:

(1) X is a subspace of $F(X)$,

(2) each continuous mapping from X to an arbitrary topological group G extends to a continuous homomorphism from $F(X)$ to G.

The *free Abelian topological group* $A(X)$ over a space X in the sense of Markov is the free algebraic Abelian group over the set X equipped with the group topology $\mathcal{T}$ having the properties (1) and (2) for an arbitrary Abelian topological group G. In this paper, we discuss only free Abelian topological groups $A(X)$, while some results obtained in this paper hold also for free topological groups $F(X)$.

We denote the *unit element* of $A(X)$ by 0. Any $g \in A(X)$ except 0 has the unique reduced representation of the form $g = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n$, where $x_i \in X$ and $\varepsilon_i = \pm 1$ for $i = 1, \ldots, n$. We put

$$l_+(g) = |\{i \leq n\colon \varepsilon_i = 1\}|, \qquad l_-(g) = |\{i \leq n\colon \varepsilon_i = -1\}|,$$

and

$$l(g) = l_+(g) + l_-(g).$$

We call the number $l(g)$ the *length* of g (by definition, $l(0) = 0$). Let $A_0 = \{g \in A(X)\colon l_+(g) = l_-(g)\}$, then A_0 is a clopen subgroup of $A(X)$, and is a clopen neighborhood of 0 in $A(X)$.

For each $n \in \mathbb{N}$, let $A_n(X) = \{g \in A(X)\colon\ l(g) \leq n\}$ (by definition $A_0(X) = \{0\}$) and define the mapping $i_n\colon (X \oplus -X \oplus \{0\})^n \to A_n(X)$ by $i_n((x_1, x_2, \ldots, x_n)) = x_1 + x_2 + \cdots + x_n$ for each $x_i \in X \oplus -X \oplus \{0\}$.

The following fundamental properties related to $A(X)$ are used often in this paper (see [1, 15]).

Lemma 1.1. (1) *Let $\mathcal{T}_1$ be a group topology for $A(X)$ which induces the original topology for X, then $\mathcal{T}_1 \leq \mathcal{T}$.*

(2) *X and $A_n(X)$, $n \in \mathbb{N}$, are closed subspaces of $A(X)$.*

(3) *The mapping i_n is continuous for each $n \in \mathbb{N}$, so that $A_n(X)$ is compact for each $n \in \mathbb{N}$ if X is compact.*

(4) *Let Y be a closed subspace of a metrizable space X, then $A(Y)$ is embedded into $A(X)$ as a closed topological subgroup. In fact, the continuous homomorphic extension of the inclusion mapping from Y to X over $A(Y)$ is a closed embedding. Thus, $A_n(Y)$ is also embedded into $A_n(X)$ as a closed subspace for each $n \in \mathbb{N}$.*

For each subset E in $A(X)$, we set

$$\operatorname{car} E = \min\{B \subset X\colon\ E \subset A(B, X)\},$$

where $A(B, X)$ is the subgroup of $A(X)$ generated by B, and it is called the *carrier* of E in X [3]. Recall that a subset E of a space X is *bounded* (in X) iff every real-valued continuous function on X is bounded on E. The following theorem was proved in [3].

Theorem 1.2. *If E is a bounded set in $A(X)$, then* car E *is bounded in X.*

A space X is a *k-space* if for each $A \subset X$, the set A is closed in X provided that the intersection of A with any compact subspace K of the space X is closed in K. Every first-countable space, and every quotient image of a k-space is a k-space. A space X is a *k_ω-space* if there is a countable cover $\mathcal{K}$ of X consisting of compact subsets of X such that the set A is closed in X provided that the intersection of A with any element K of $\mathcal{K}$ is closed in K. It is known that a metrizable space X is a k_ω-space iff X is locally compact and separable [17, Proposition 8.5]. Some characterizations of that $A(X)$ is a k_ω- or k-space were obtained by [3, 10].

2. Fundamental results

In this section, we will construct a neighborhood base $\mathcal{W}$ of 0 in $A(X)$. The idea of our construction is due to the one of the neighborhood base Σ^* of the unit element in $F(X)$ by Tkačenko [16]. He used the universal uniformity on $(X \oplus X^{-1} \oplus \{e\})^n$ for each $n \in \mathbb{N}$ to construct the neighborhood base Σ^*. On the other hand, we used only the universal uniformity $\mathcal{U}_X$ on X for $A(X)$. The neighborhood base $\mathcal{W}$ of 0 in $A(X)$ can be obtained from the neighborhood base of e in $F(X)$ by Pestov [13]. Nevertheless, since he gave it without the proof and the construction of $\mathcal{W}$ is important in this paper, we give its construction and proof here. Furthermore, we will construct a neighborhood base $\mathcal{W}_n$ of 0 in $A_{2n}(X)$ for each $n \in \mathbb{N}$. The existence of the neighborhood base $\mathcal{W}_n$ is one of the important facts in this paper.

Constructing the neighborhood bases, we introduce some notations. Let $(X, \mathcal{U})$ be a uniform space. The *inverse relation* of $U \in \mathcal{U}$ will be denoted by U^{-1}, and the *composition* of U and V in $\mathcal{U}$ will be denoted by $U \circ V$; thus we have

$$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$$

and

$$U \circ V = \{(x, z) \in X \times X : \text{there is a } y \in X \text{ such that } (x, y) \in U \text{ and } (y, z) \in V\}.$$

The *diagonal* of $X \times X$ is the set $\Delta_X = \{(x, x) : x \in X\}$. A set $U \in \mathcal{U}$ is called *symmetric* if $U = U^{-1}$.

Lemma 2.1. *Let $k \in \mathbb{N} \cup \{0\}$, $p, k_1, \ldots, k_p \in \mathbb{N}$ such that $\sum_{i=1}^{p} 2^{-k_i} < 2^{-k}$.*

(1) *Let $(X, \mathcal{U})$ be a uniform space and $\{U_n : n \in \mathbb{N} \cup \{0\}\}$ a countable subcollection of $\mathcal{U}$ such that $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n$ for each $n \in \mathbb{N} \cup \{0\}$, then $U_{k_1} \circ U_{k_2} \circ \cdots \circ U_{k_p} \subset U_k$.*

(2) *Let G be a group with the unit element e and $\{V_n : n \in \mathbb{N} \cup \{0\}\}$ a countable collection of subsets of G such that $e \in V_n$ and $V_{n+1} \cdot V_{n+1} \cdot V_{n+1} \subset V_n$ for each $n \in \mathbb{N} \cup \{0\}$, then $V_{k_1} \cdot V_{k_2} \cdots V_{k_p} \subset V_k$* [15].

Proof. Since the proof is straightforward, we only give an outline of the proof of (1). We prove by induction with respect to p. Assume that for each $n \leqslant p$, condition (1) is obtained. If there is $j \in \{1, 2, \ldots, p+1\}$ such that $k_j = k+1$, by the inductive assumption,

$$U_{k_1} \circ U_{k_2} \circ \cdots \circ U_{k_{p+1}} \subset U_{k+1} \circ U_{k+1} \circ U_{k+1} \subset U_k.$$

Thus, let $k_j < k+1$ for each $j \in \{1, 2, \ldots, p+1\}$. In this case, if $\sum_{i=1}^{p+1} 2^{-k_i} < 2^{-(k+1)}$, we can show that

$$U_{k_1} \circ U_{k_2} \circ \cdots \circ U_{k_p} \subset U_{k+1},$$

therefore

$$U_{k_1} \circ U_{k_2} \circ \cdots \circ U_{k_{p+1}} \subset U_{k+1} \circ U_{k+1} \subset U_k.$$

Otherwise, i.e., $2^{-(k+1)} \leqslant \sum_{i=1}^{p} 2^{-k_i} < 2^{-k}$, then there is $j \in \{2, \ldots, p\}$ such that

$$\sum_{i=1}^{j} 2^{-k_i} < 2^{-(k+1)} \quad \text{and} \quad \sum_{i=1}^{j+1} 2^{-k_i} \geqslant 2^{-(k+1)}.$$

It follows that

$$U_{k_1} \circ U_{k_2} \circ \cdots \circ U_{k_{p+1}} \subset U_{k+1} \circ U_{k_{j+1}} \circ U_{k+1} \subset U_{k+1} \circ U_{k+1} \circ U_{k+1} \subset U_k.$$

Consequently, condition (1) is obtained. □

Let $\mathscr{U}_X$ be the universal uniformity on a space X and put $\mathscr{P} = \{P \subset \mathscr{U}_X : P$ is countable$\}$. For each $P = \{U_1, U_2, \ldots\} \in \mathscr{P}$, let

$$W(P) = \{x_1 - y_1 + x_2 - y_2 + \cdots + x_k - y_k : (x_i, y_i) \in U_i \text{ for } i = 1, 2, \ldots, k,\ k \in \mathbb{N}\},$$

and

$$\mathscr{W} = \{W(P) : P \in \mathscr{P}\}.$$

Furthermore, fix any $n \in \mathbb{N}$. Let

$$\mathscr{Q}_n(P) = \{Q \subset P : |Q| = n\},$$

$$W_n(P) = \{x_1 - y_1 + x_2 - y_2 + \cdots + x_n - y_n : (x_j, y_j) \in U_{i_j} \text{ for } j = 1, 2, \ldots, n,\ \{U_{i_1}, U_{i_2}, \ldots, U_{i_n}\} \in \mathscr{Q}_n(P)\},$$

and

$$\mathscr{W}_n = \{W_n(P) : P \in \mathscr{P}\}.$$

Remark 2.2. In the above definition, for $P \in \mathscr{P}$, there may be the same elements in P. In particular, for each $U \in \mathscr{U}_X$, the countable collection $\{U, U, \ldots\}$ is also in $\mathscr{P}$.

The reader should remark that the representation of the element of $W(P)$ and $W_n(P)$ need not be a reduced representation.

For the definition of $W_n(P)$, let

$$\mathcal{R}_n(P)=\{Q\subset P: |Q|\leq n\}.$$

Since, Δ_X is contained in each $U\in\mathcal{U}_X$, it is easy to see that

$$W_n(P)=\{x_1-y_1+x_2-y_2+\cdots+x_k-y_k : (x_j, y_j)\in U_{i_j} \text{ for } j=1,2,\ldots,k,\\ \{U_{i_1}, U_{i_2},\ldots, U_{i_k}\}\in\mathcal{R}_n(P)\}.$$

Theorem 2.3. *$\mathcal{W}$ is a neighborhood base of 0 in $A(X)$.*

Proof. First, we shall show that $\mathcal{W}$ satisfies the axioms for open sets in the Abelian topological group $A(X)$, i.e., $\mathcal{W}$ satisfies the following properties:

(i) for every $V\in\mathcal{W}$, there is a $W\in\mathcal{W}$ such that $W+W\subset V$;
(ii) for every $V\in\mathcal{W}$, there is a $W\in\mathcal{W}$ such that $-W\subset V$;
(iii) for every $V\in\mathcal{W}$ and every $g\in V$, there is a $W\in\mathcal{W}$ such that $g+W\subset V$;
(iv) for every U, $V\in\mathcal{W}$, there is a $W\in\mathcal{W}$ such that $W\subset U\cap V$;
(v) $\{0\}=\bigcap\mathcal{W}$.

Let $P=\{U_1, U_2,\ldots\}\in\mathcal{P}$ and $g\in W(P)$. Assume that $g=x_1-y_1+x_2-y_2+\cdots+x_n-y_n$ such that $(x_i, y_i)\in U_i$ for $i=1,2,\ldots,n$, for some $n\in\mathbb{N}$. Take $P_1=\{A_1, A_2,\ldots\}$, $P_2=\{B_1, B_2,\ldots\}$ and $P_3=\{C_1, C_2,\ldots\}$ such that

(1) $P_1, P_2, P_3\in\mathcal{P}$,
(2) $A_i\subset U_{2i-1}\cap U_{2i}$ for each $i\in\mathbb{N}$,
(3) $B_i\subset U_i$ and B_i is symmetric for each $i\in\mathbb{N}$,
(4) $C_i\subset U_{i+n}$ for each $i\in\mathbb{N}$.

Then it can be shown that $W(P_1)+W(P_1)\subset W(P)$, $-W(P_2)\subset W(P)$, and $g+W(P_4)\subset W(P)$. These imply that the conditions (i), (ii), and (iii) hold. The conditions (iv) and (v) are easily seen, so that we omit the proof.

Thus, let $\mathcal{T}_1$ be a group topology for $A(X)$ generated by $\mathcal{W}$. Take $P=\{U_1, U_2,\ldots\}\in\mathcal{P}$ and $x\in X$, and put $W(x)=\{y\in X: (y,x)\in U_1\}$. Then, since $\mathcal{U}_X$ is compatible with the original topology for X, $W(x)$ is open in X. Also, we can show that $x\in W(x)\subset(W(P)+x)\cap X$, and this means that $\mathcal{T}_1|_X$ is coarser than the original topology for X.

Claim. *$\mathcal{T}_1$ is finer than the topology of $A(X)$.*

Proof of Claim. Let V be an open neighborhood of 0 in $A(X)$. Put $V_0=V$ and take a sequence $\{V_n : n\in\mathbb{N}\}$ of neighborhoods of 0 in $A(X)$ such that $V_n+V_n+V_n\subset V_{n-1}$ for each $n\in\mathbb{N}$. Let $U_n=\{(x,y)\in X\times X: x-y\in V_n\}$ for each $n\in\mathbb{N}$, and $P=\{U_1, U_2,\ldots\}$. Since $U_n\in\mathcal{U}_X$ for each $n\in\mathbb{N}$, $P\in\mathcal{P}$. Take any point $g\in W(P)$, then there is an $n\in\mathbb{N}$ such that

$$g=x_1-y_1+\cdots+x_n-y_n$$

for some $(x_i, y_i) \in U_n$ for $i = 1, 2, \ldots, n$. Thus, by Lemma 2.1(2),

$$g \in V_1 + V_2 + \cdots + V_n \subset V_0 = V.$$

It follows that $W(P) \subset V$.

By this claim, $\mathcal{T}_1|_X$ coincides with the original topology for X. Thus, by Lemma 1.1(1), $\mathcal{T}_1$ is coarser than the topology for $A(X)$. Consequently, $\mathcal{T}_1$ coincides with the topology for $A(X)$, and $\mathcal{W}$ is a neighborhood base of 0 in $A(X)$. □

Theorem 2.4. *$\mathcal{W}_n$ is a neighborhood base of* 0 *in $A_{2n}(X)$ for each $n \in \mathbb{N}$.*

Proof. Fix any $n \in \mathbb{N}$ and fix it. By Theorem 2.3, $\mathcal{W}|_{A_{2n}(X)} = \{W(P) \cap A_{2n}(X): P \in \mathcal{P}\}$ is a neighborhood base of 0 in $A_{2n}(X)$. For each $P = \{U_1, U_2, \ldots\} \in \mathcal{P}$, it is clear that $W_n(P) \subset W(P) \cap A_{2n}(X)$. Thus, to complete the proof, it suffices to show the following claim.

Claim. *For each $P \in \mathcal{P}$, there is a $P_1 \in \mathcal{P}$ such that $W(P_1) \cap A_{2n}(X) \subset W_n(P)$.*

Proof of Claim. Let $P = \{U_1, U_2, \ldots\} \in \mathcal{P}$. Put $V_0 = U_1$ and inductively take a collection $\{V_m : m \in \mathbb{N}\} \subset \mathcal{U}_X$ such that

$$V_m \circ V_m \circ V_m \subset V_{m-1} \cap U_{m+1} \quad \text{for each } m \in \mathbb{N}.$$

For the collection $P_1 = \{V_m : m \in \mathbb{N}\}$, we shall show that $W(P_1) \cap A_{2n}(X) \subset W_n(P)$. Take any $g \in W(P_1) \cap A_{2n}(X)$, and let

$$g = x_1 - y_1 + x_2 - y_2 + \cdots + x_k - y_k, \tag{1}$$

where $(x_i, y_i) \in V_i$ for $i = 1, 2, \ldots, k$ and $k \in \mathbb{N}$. Now, we put

$$A(g) = \{x_i : x_i \text{ is not reduced in the representation (1) of } g,\ i = 1, 2, \ldots, k\},$$

and

$$B(g) = \{y_i : y_i \text{ is not reduced in the representation (1) of } g,\ i = 1, 2, \ldots, k\}.$$

If there are $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$ such that $x_i = x_j$ $(y_i = y_j)$, we regard that these elements x_i and x_j (y_i and y_j) are different elements in $A(g)$ ($B(g)$), respectively. Since $g \in A_{2n}(X)$, we can see that $|A(g)| = |B(g)|$ and $|A(g)| + |B(g)| \leq 2n$. Thus, we put $A(g) = \{a_1, a_2, \ldots, a_l\}$ for some $l \leq n$, and take $a_i \in A(g)$. Then there is a $k(i, 1) \in \{1, 2, \ldots, k\}$ such that $a_i = x_{k(i,1)}$. If $y_{k(i,1)} \in B(g)$, we put $b_{\varphi(i)} = y_{k(i,1)}$. Otherwise, $y_{k(i,1)}$ is reduced in the representation (1) of g, so that there is a $k(i, 2) \in \{1, \ldots, k\}$ such that $y_{k(i,1)} = x_{k(i,2)}$, and clearly, $k(i, 2) \neq k(i, 1)$. If $y_{k(i,2)} \in B(g)$, we put $b_{\varphi(i)} = y_{k(i,2)}$. Otherwise, in the same way, take a $k(i, 3) \in \{1, 2, \ldots, k\}$ such that $y_{k(i,2)} = x_{k(i,3)}$, and $k(i, 3) \notin \{k(i, 1), k(i, 2)\}$. We continue this process till an element of $B(g)$ appears and denote the element of $B(g)$ by $b_{\varphi(i)}$. Clearly, the element $b_{\varphi(i)}$ must be appeared. Furthermore, we carry out this work for every element of $A(g)$. Thus, we get a permutation φ on $\{1, 2, \ldots, l\}$ and sequences $\{k(i, 1), k(i, 2), \ldots, k(i, j(i))\} \subset \{1, 2, \ldots, k\}$, $i = 1, 2, \ldots, l$, such that

$$a_i = x_{(i,1)} \quad \text{and} \quad b_{\varphi(i)} = y_{k(i,j(i))}, \quad \text{for } i = 1, 2, \ldots, l, \tag{2}$$

$$y_{k(i,j)} = x_{k(i,j+1)} \quad \text{for } j = 1, 2, \ldots, j(i) - 1,\ i = 1, 2, \ldots, l, \tag{3}$$

and

$$\{k_{(i,j)}: j=1,2,\ldots,j(i),\ i=1,2,\ldots,l\}$$
$$\text{consists of distinct numbers of } \{1,2,\ldots,k\}. \tag{4}$$

Thus, from (2) and (3), we have for each $i=1,2,\ldots,l$,

$$(a_i, b_{\varphi(i)}) \in V_{k(i,1)} \circ V_{k(i,2)} \circ \cdots \circ V_{k(i,j(i))}.$$

Now, let $k(i)=\min\{k(i,1), k(i,2),\ldots,k(i,j(i))\}$, then by (4), the sequence $\{k(1), k(2),\ldots,k(l)\}$ is a subsequence of $\{1,2,\ldots,n\}$ consisting of distinct elements. Thus, by Lemma 2.1(1) and the definition of P_1,

$$(a_i, b_{\varphi(i)}) \in V_{k(i)-1} \subset U_{k(i)} \quad \text{for each } i=1,2,\ldots,l.$$

On the other hand, since φ is a permutation on $\{1,2,\ldots,l\}$,

$$g=a_1-b_{\varphi(1)}+a_2-b_{\varphi(2)}+\cdots+a_l-b_{\varphi(l)}.$$

Consequently, since $\{U_{k(1)}, U_{k(2)},\ldots,U_{k(l)}\} \in \mathcal{R}_n(P)$, by Remark 2.2, we have $g \in W_n(P)$, so that $W(P_1) \cap A_{2n}(X) \subset W_n(P)$. □

For a space X and each $n \in \mathbb{N}$, we define a mapping j_n from X^{2n} $(=X^n \times X^n)$ to $A_{2n}(X)$ as follows

$$j_n((x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n))=x_1+x_2+\cdots+x_n-(y_1+y_2+\cdots+y_n)$$

for each $(x_1,x_2,\ldots,x_n)$ and $(y_1,y_2,\ldots,y_n) \in X^n$. Theorem 2.4 gives a following important result, which is often used to prove our main theorems.

Corollary 2.5. *Let X be a space, $n \in \mathbb{N}$ and E be a subset of $A_{2n}(X)$. Then, $0 \in \bar{E}$ if and only if $j_n^{-1}(E) \cap U^n \neq \emptyset$ for each $U \in \mathcal{U}_X$, where $U^n=\{((x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n)) \in X^{2n}: (x_i,y_i) \in U,\ i=1,2,\ldots,n\}$.*

Proof. ($\Rightarrow$) Let $U \in \mathcal{U}_X$ and put $P=\{U_1,U_2,\ldots\} \in \mathcal{P}$ such that $U_i=U$ for each $i \in \mathbb{N}$. Since $W_n(P)$ is a neighborhood of 0 in $A_{2n}(X)$, we can take a $g \in W_n(P) \cap E$. Then, we have

$$g=x_1-y_1+x_2-y_2+\cdots+x_n-y_n,$$

where $(x_i,y_i) \in U_i$ for $i=1,2,\ldots,n$. Thus, for $x=((x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n))$, it is clear that $x \in j_n^{-1}(E) \cap U^n$.

($\Leftarrow$) Let $P=\{U_1,U_2,\ldots\} \in \mathcal{P}$, and take $U \in \mathcal{U}_X$ such that $U \subset U_1 \cap U_2 \cap \cdots \cap U_n$. By the assumption, we can take $x=((x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n)) \in j_n^{-1}(E) \cap U^n$. Since, $(x_i,y_i) \in U \subset U_i$ for $i=1,2,\ldots,n$, it follows that $j_n(x) \in W_n(P) \cap E$. Therefore, by Theorem 2.4, $0 \in \bar{E}$. □

Corollary 2.6. *Let X be a paracompact space and E a subset of $A_2(X)$. Then, $0 \in \bar{E}$ if and only if $j_1^{-1}(E) \cap \Delta_X \neq \emptyset$.*

Proof. Since X is paracompact, every open neighborhood U of Δ_X in X^2 is contained in $\mathcal{U}_X$. Thus, by Corollary 2.5, it is obvious. □

3. Test spaces

In this section, we introduce three spaces M_1, M_2 and M_3 which have the following properties:

(1) $A_n(M_1)$ is a k-space for each $n \in \mathbb{N}$,

(2) $A_3(M_2)$ is not a k-space, and

(3) $A_4(M_3)$ is not a k-space.

The constructions of these spaces are simple, but they play essential roles in the proof of the main results in this paper.

Constructions. Let M_1 be a metrizable space such that $M_1 = X_0 \cup \bigcup_{i=1}^{\infty} X_i$ such that

(1) X_i is an infinite discrete open subspace of M_1 for each $i \in \mathbb{N}$, and

(2) X_0 is a compact subspace of M_1 and $\{V_k = X_0 \cup \bigcup_{i=k}^{\infty} X_i : k \in \mathbb{N}\}$ is a neighborhood base of X_0 in X, i.e., for each open set U in X which contains X_0, there is a $k \in \mathbb{N}$ such that $X_0 \subset V_k \subset U$.

In the above definition, if each X_i consists of countably many elements and X_0 is a one-point set, we denote the space by M_1'. We put $C = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ with the subspace topology of I. Let $M_2 = \bigoplus\{C_i : i \in \mathbb{N}\} \oplus M_1'$, where C_i is a copy of C for each $i \in \mathbb{N}$. Let $M_3 = \bigoplus\{C_\alpha : \alpha < \omega_1\}$, where C_α is a copy of C for each $\alpha < \omega_1$.

Remark 3.1. The concrete example of M_1 is the hedgehog space $J(\kappa)$ of spininess κ such that each spininess is a sequence which converges to the center point. In [18] we proved that $A_n(J(\kappa))$ is a k-space for each $n \in \mathbb{N}$. On the other hand, it was proved in [3] that $A(J(\kappa))$ is not a k-space if $\kappa \geq \aleph_0$. Furthermore, by [18], we know that $A_4(M_2)$ is not a k-space.

Theorem 3.2. *$A_n(M_1)$ is a k-space for each $n \in \mathbb{N}$.*

Proof. In order to prove the theorem, it suffices to show that

$$\text{for each } n \in \mathbb{N} \text{ and } E \subset A_n(M_1) \text{ such that } E \cap K \text{ is closed in } K \text{ for each compact subset } K \text{ of } A_n(M_1), \text{ if } 0 \in \bar{E} \text{ then } 0 \in E. \qquad (*)$$

For, if $A_n(M_1)$ is not a k-space for some $n \in \mathbb{N}$, then there is a subset H of $A_n(M_1)$ such that $H \cap K$ is closed in K for each compact subset K of $A_n(M_1)$ and $\bar{H} \setminus H \neq \emptyset$. Take a point $g \in \bar{H} \setminus H$, and let $E = H - g$. Then, it can be seen that E is a subset of $A_{2n}(M_1)$ such that $E \cap K$ is closed in K for each compact subset K of $A_{2n}(M_1)$ and $0 \in \bar{E} \setminus E$.

Thus, let prove the property $(*)$. Take an arbitrary $n \in \mathbb{N}$ and fix it (we can assume that $n \geq 2$). Let E be a subset of $A_n(M_1)$ such that $E \cap K$ is closed in K for each compact subset K of $A_n(M_1)$, and assume that $0 \in \bar{E}$. Since A_0 (see Section 1) is an open neighborhood of 0 in $A(M_1)$, $0 \in \overline{E \cap A_0}$. Furthermore, note that

$$E \cap A_0 = \bigcup \left\{ E \cap (A_{2m}(M_1) \backslash A_{2m-1}(M_1)) \colon m \leq \frac{n}{2},\ m \in \mathbb{N} \right\},$$

then there is an $m \in \mathbb{N}$ with $m \leq n/2$ such that

$$0 \in \overline{E \cap (A_{2m}(M_1) \backslash A_{2m-1}(M_1))}.$$

And, we put $D = E \cap (A_{2m}(M_1) \backslash A_{2m-1}(M_1))$.

On the other hand, by the properties (1) and (2) of the definition of M_1, we can find a countable uniform base $\mathscr{U}$ of the universal uniformity $\mathscr{U}_{M_1}$ of M_1 such that $\mathscr{U} = \{U_k = G_k \cup \Delta_{M_1} \colon k \in \mathbb{N}\}$, where for each $k \in \mathbb{N}$, G_k is an open neighborhood of Δ_{X_0} in $M_1 \times M_1$ such that $G_k \subset V_k \times V_k$. Now, apply Corollary 2.5, then we have

$$j_m^{-1}(D) \cap (U_k)^m \neq \emptyset \quad \text{for each } k \in \mathbb{N}.$$

Let us take a point $x_k \in j_m^{-1}(D) \cap (U_k)^m$ for each $k \in \mathbb{N}$. Since $g_k = j_m(x_k) \in A_{2m}(M_1) \backslash A_{2m-1}(M_1)$, $x_k \in (G_k)^m$, and $\operatorname{car} g_k \subset V_k$. It follows that $K = \bigcup \{\operatorname{car} g_k \colon k \in \mathbb{N}\} \cup X_0$ is a compact subset of M_1, and by Lemma 1.1(3) and (4), $A_n(K)$ is a compact subset of $A_n(M_1)$. Hence we have $E \cap A_n(K)$ is closed in $A_n(K)$. Since $\{x_i \colon i \in \mathbb{N}\} \subset j_m^{-1}(D)$,

$$\{g_i \colon i \in \mathbb{N}\} \subset D \cap A_n(K) \subset E \cap A_n(K).$$

On the other hand,

$$\{x_i \colon i \in \mathbb{N}\} \cap (U_k)^m \neq \emptyset \quad \text{for each } k \in \mathbb{N}.$$

Hence, by Corollary 2.5, $0 \in \overline{\{g_i \colon i \in \mathbb{N}\}}$. Thus we have

$$0 \in \overline{E \cap A_n(K)} = E \cap A_n(K) \subset E.$$

Consequently, $A_n(M_1)$ is a k-space for each $n \in \mathbb{N}$. $\square$

Corollary 3.3. *$A_n(J(\kappa))$ is a k-space for each $n \in \mathbb{N}$.*

Theorem 3.4. *$A_3(M_2)$ is not a k-space.*

Proof. For each $n \in \mathbb{N}$, we put $X_n = \{x_{n,i} \colon i \in \mathbb{N}\}$, $C_n = \{c_{n,i} \colon i \in \mathbb{N}\} \cup \{c_n\}$, and $X_0 = \{x\}$. For each $n, j \in \mathbb{N}$, let

$$g_{n,j} = c_n - c_{n,j} + x_{n,j}, \quad \text{and} \quad E = \{g_{n,j} \colon n, j \in \mathbb{N}\}.$$

We shall prove that

(1) $E \cap K$ is closed in K for each compact subset K in $A_3(M_2)$, and

(2) $x \in \bar{E} \backslash E$.

Let K be a compact subset of $A_3(M_2)$, in fact of $A(M_2)$. Then, by Theorem 1.2, car K is bounded in M_2. Hence, there are finite subsets F_1 and $F_{2(n)}$ of $\mathbb{N}$, $n \in \mathbb{N}$, such that

$$\operatorname{car} K \subset \bigcup \{C_n : n \in F_1\} \cup \bigcup \{x_{n,i} : i \in F_{2(n)},\ n \in \mathbb{N}\} \cup \{x\}.$$

Thus, by the definition of E,

$$\operatorname{car}(K \cap E) \subset \bigcup \{\operatorname{car} g_{n,i} : n \in F_1 \text{ and } i \in F_{2(n)}\}.$$

Hence, $K \cap E$ is finite and thereby it is closed in K.

Next, we shall prove (2). Since $x \notin E$, we shall show that $x \in \bar{E}$. Since $A_3(M_2)$ is closed in $A(M_2)$, it suffices to show that $x \in \bar{E}^{A(M_2)}$. Let U be an open neighborhood of x in $A(M_2)$. Then we can choose an open neighborhood W of 0 in $A(M_2)$ such that $W + W + x \subset U$. Since $W + x$ is an open neighborhood of x in $A(M_2)$, there is an $N \in \mathbb{N}$ such that $\{x_{n,i} : n \geqslant N,\ i \in \mathbb{N}\} \subset W + x$. And, for each $n \geqslant N$, there is an $i_n \in \mathbb{N}$ such that $c_n - c_{n,i_n} \in W$. It follows that

$$\{g_{n,i_n} : n \geqslant N\} \subset W + W + x \subset U.$$

Thus $x \in \bar{E}$. $\square$

To prove that $A_4(M_3)$ is not a k-space, we need the following technical lemma which was obtained in [9].

Lemma 3.5. *There is a collection* $\mathscr{E} = \{E_\alpha : \alpha < \omega_1\}$ *of infinite subsets of* $\mathbb{N}$ *such that*

(1) $E_\alpha \cap E_\beta$ *is finite for each* α, $\beta < \omega_1$ *with* $\alpha \neq \beta$,

(2) *for each* $\alpha < \omega_1$, $\max(E_\alpha \cap E_\beta) \neq \max(E_\alpha \cap E_\gamma)$ *for each* β, $\gamma < \alpha$ *with* $\beta \neq \gamma$.

Theorem 3.6. $A_4(M_3)$ *is not a k-space.*

Proof. Let $\mathscr{A} = \{A_\alpha : \alpha < \omega_1\}$ and $\mathscr{B} = \{B_\beta : \beta < \omega_1\}$ are uncountable subcollections of $\mathscr{E}$ defined in Lemma 3.5 such that $\mathscr{A} \cup \mathscr{B} = \mathscr{E}$ and $\mathscr{A} \cap \mathscr{B} = \emptyset$. Then it can be proved that

(3) there are no infinite subsets A of $\mathbb{N}$ such that $A_\alpha \setminus A$ is finite and $B_\alpha \cap A$ is finite for each $\alpha < \omega_1$.

We put $C_\alpha = \{x_{\alpha,n} : n \in \mathbb{N}\} \cup \{x_\alpha\}$, where x_α is the limit point of $\{x_{\alpha,n} : n \in \mathbb{N}\}$, for each $\alpha < \omega_1$. Let, for each $\alpha, \beta < \omega_1$,

$$A_{\alpha,\beta} = \{x_{\alpha,n} - x_\alpha + x_{\beta,n} - x_\beta : n \in A_\alpha \cap B_\beta\},$$

and

$$E = \bigcup \{A_{\alpha,\beta} : \alpha, \beta < \omega_1\}.$$

We shall show that

(4) $E \cap K$ is closed in K for each compact subset K of $A_4(M_3)$, and

(5) $0 \in \bar{E} \setminus E$.

Let K be a compact subset of $A_4(M_3)$, then car K is bounded in M_3 by Theorem 1.2. We can take a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of ω_1 such that car $K \subset \bigcup_{i=1}^{n} C_{\alpha_i}$, so that $E \cap K \subset \bigcup \{A_{\alpha_i,\alpha_j} : i, j = 1, 2, \ldots, n\}$. On the other hand, by property (1) of Lemma 3.5, each $A_{\alpha,\beta}$ is finite. It follows that $E \cap K$ is finite, and (4) is proved.

Before we show (5), we shall define a uniform base $\mathcal{U}$ of the universal uniformity on M_3, as follows. For each $\alpha < \omega_1$ and $k \in \mathbb{N}$, let $V_{\alpha,k} = \{x_{\alpha,m} : m \geq k\} \cup \{x_\alpha\}$, and we put

$$\mathcal{U}_\alpha = \{U_{\alpha,k} = (V_{\alpha,k} \times V_{\alpha,k}) \cup \Delta_\alpha : k \in \mathbb{N}\},$$

where Δ_α is the diagonal of $C_\alpha \times C_\alpha$. For each $M = \{n_\alpha \in \mathbb{N}: \alpha < \omega_1\} \in \mathbb{N}^{\omega_1}$, let

$$U(M) = \bigcup \{U_{\alpha,n_\alpha} : \alpha < \omega_1\}, \quad \text{and} \quad \mathcal{U} = \{U(M): M \in \mathbb{N}^{\omega_1}\}.$$

Then, $\mathcal{U}$ is a uniform base of the universal uniformity on M_3. To show that $0 \in \bar{E}$, by Corollary 2.5, it suffices to show that

$$j_2^{-1}(E) \cap U^2 \neq \emptyset \quad \text{for each } U \in \mathcal{U}.$$

Take an $M = \{n_\alpha : \alpha < \omega_1\} \in \mathbb{N}^{\omega_1}$, and for each $\alpha < \omega_1$, let

$$A'_\alpha = \{n \in A_\alpha : n \geq n_\alpha\} \quad \text{and} \quad B'_\alpha = \{n \in B_\alpha : n \geq n_\alpha\}.$$

Clearly each A'_α is an infinite subset of $\mathbb{N}$. Assume that $A'_\alpha \cap B'_\beta = \emptyset$ for each α, $\beta < \omega_1$. Then $\bigcup_{\alpha<\omega_1} A'_\alpha \cap \bigcup_{\beta<\omega_1} B_\beta = \emptyset$. For the infinite set $A = \bigcup_{\alpha<\omega_1} A_\alpha$ in $\mathbb{N}$, we can prove that

$$A \cap B_\beta \subset \{N \in \mathbb{N}: n < n_\beta\} \quad \text{for each } \beta < \omega_1.$$

Hence $A \cap B_\beta$ is finite for each $\beta < \omega_1$, and also

$$A_\alpha \backslash A \subset A_\alpha \backslash A'_\alpha = \{n \in \mathbb{N}: n < n_\alpha\}.$$

Thereby $A_\alpha \backslash A$ is finite for each $\alpha < \omega_1$. This contradicts the property (3). Thus, there are α, $\beta < \omega_1$ such that $A'_\alpha \cap B'_\beta \neq \emptyset$. Take an $n \in A'_\alpha \cap B'_\beta$, then

$$(x_{\alpha,n}, x_\alpha) \in V_{\alpha,n_\alpha} \times V_{\alpha,n_\alpha} \subset U(M),$$

and

$$(x_{\beta,n,} x_\beta) \in V_{\beta,n_\beta} \times V_{\beta,n_\beta} \subset U(M).$$

It follows that

$$x = ((x_{\alpha,n}, x_{\beta,n}), (x_\alpha, x_\beta)) \subset U(M) \times U(M).$$

Since $j_2(x) = x_{\alpha,n} - x_\alpha + x_{\beta,n} - x_\beta \in E$,

$$x \in j_2^{-1}(E) \cap U(M)^2 \neq \emptyset.$$

Consequently, $A_4(M_3)$ is not a k-space. □

Remark 3.7. The essential idea of the proof of (5) is used by Malyhin to prove that the tightness of the product $V(\aleph_1)\times V(\aleph_1)$ is ω_1. The proof was cited with his permission in [3]. Therefore, $V(\aleph_1)\times V(\aleph_1)$ is not a k-space. Since X^2 is closed embedded in $A_3(X)$ for each space X (cf. [14]), $A_3(V(\aleph_1))$ is not a k-space. In fact, in Section 4, we shall show that $A_2(V(\aleph_1))$ is not a k-space.

4. Main theorems

In this section, we shall give characterizations of a metrizable space X such that every $A_n(X)$ is a k-space for $n\geqslant 4$, $A_3(X)$ is a k-space, and $A_2(X)$ is a k-space, respectively. First, we introduce the result for the mapping i_n, which was pointed out in [2]. Recall that a space X is *Dieudonné complete* if there is a complete uniformity on the space X [4]. Every paracompact space is Dieudonné complete, and the closure of every bounded subset of a Dieudonné complete space is compact.

Proposition 4.1. *Let X be a Dieudonné complete space. Then, for each $n\in\mathbb{N}$, the mapping i_n is quotient if $A_n(X)$ is a k-space.*

Its proof can be easily obtained from the following well-known facts, and therefore we omit the proof.

(1) *For each compact subset K of $A_n(X)$, there is a compact subset C of $(X\oplus -X\oplus \{0\})^n$ such that $i_n^{-1}(K)\subset C$ if X is a Dieudonné complete space* [3].

(2) *A continuous mapping $f: X\to Y$ of a topological space to a k-space Y is quotient if and only if for every compact subset $Z\subset Y$ the restriction $f|_{f^{-1}(Z)}: f^{-1}(Z)\to Z$ is quotient* [9, Theorem 3.3.22].

Theorem 4.2. *If X is a metrizable space, then the following statements are equivalent:*

(a) *$A_n(X)$ is a k-space for each $n\in\mathbb{N}$,*

(b) *$A_4(X)$ is a k-space,*

(c) *i_n is a quotient mapping for each $n\in\mathbb{N}$,*

(d) *i_4 is a quotient mapping,*

(e) *either X is locally compact and the set X' of all nonisolated points of X is separable, or X' is compact.*

Proof. The implications (a)$\Rightarrow$(b) and (c)$\Rightarrow$(d) are clear, and by Proposition 4.1, (a) and (c), (b) and (d) are equivalent. Thus, it suffices to show the implications (b)$\Rightarrow$(e) and (e)$\Rightarrow$(a).

(b)$\Rightarrow$(e) Assuming the contrary, we can consider the following two cases.

Case 1: *X is not locally compact and X' is not compact.*

Case 2: *X' is not separable.*

In Case 1, we can take an infinite discrete sequence $\{c_n\in X': n\in\mathbb{N}\}$ in X. For each $n\in\mathbb{N}$, since $c_n\in X'$, there is a convergent sequence $\{c_{n,i}: i\in\mathbb{N}\}$ in X which converges to c_n. Furthermore, since X is not locally compact, we can find a point $x \in X$ and $\{B_n : n \in \mathbb{N}\}$ be a countable neighborhood base of x in X such that for each $n \in \mathbb{N}$, $B_n \supset B_{n+1}$ and $B_n \setminus B_{n+1}$ contains an infinite discrete (in X) sequence $X_n = \{x_{n,i} : i \in \mathbb{N}\}$. Without loss of generality, we may assume that the collection $\{B_1\} \cup \{C_n : n \in \mathbb{N}\}$ is discrete in X, where $C_n = \{c_{n,i} : i \in \mathbb{N}\} \cup \{c_n\}$. It follows that $Y = \bigcup\{C_n : n \in \mathbb{N}\} \cup \bigcup\{X_n : n \in \mathbb{N}\} \cup \{x\}$ is a closed subset of X. Thus Y is homeomorphic to M_2, and therefore, by Theorem 3.4, $A_3(M_2)$ is not a k-space. Hence $A_3(Y)$ is not a k-space. Since, by Lemma 1.1(4), $A_3(Y)$ is embedded into $A_3(X)$ as a closed subspace, $A_3(X)$ is not a k-space. Thus, $A_4(X)$ is not a k-space.

In Case 2, since X' has an uncountable discrete collection in X, we can take a closed subspace Y of X which is homeomorphic to M_3. Thus, by Theorem 3.6 and the same way as Case 1, we can prove that $A_4(X)$ is not a k-space.

(e)$\Rightarrow$(a) By Theorem A, if X is locally compact and X' is separable, $A(X)$ is a k-space. Since each $A_n(X)$ is closed in $A(X)$, $A_n(X)$ is a k-space for each $n \in \mathbb{N}$. Next, suppose that X' is compact and X is not compact because $A(Y)$ of a compact space Y is a k-space. Then the space X is a space of type M_1. It follows, by Theorem 3.2, that $A_n(X)$ is a k-space for each $n \in \mathbb{N}$.

Consequently, we have the theorem. □

Corollary 4.3 (Stability theorem). *For a metrizable space X, every $A_n(X)$, $n \in \mathbb{N}$, is a k-space if and only if $A_4(X)$ is a k-space.*

Case 1 of the proof of the implication (b)$\Rightarrow$(e) in Theorem 4.2 yields the following (cf. [5]).

Corollary 4.4. *Let $\mathbb{Q}$ be the space of rationals and $\mathbb{P}$ the space of irrationals. Then, neither $A_3(\mathbb{Q})$ nor $A_3(\mathbb{P})$ is a k-space.*

The following lemma can be proved from General assertion 1 in [1] or Fundamental Lemma in [8], and we omit the proof.

Lemma 4.5. *For a space X and each $n \in \mathbb{N}$, the restriction*

$$i_n | i_n^{-1}(A_n(X) \setminus A_{n-1}(X)) : i_n^{-1}(A_n(X) \setminus A_{n-1}(X)) \to A_n(X) \setminus A_{n-1}(X)$$

of i_n is an open and closed $n!$-to-1 mapping.

The following theorem can be proved from the result in [13]. Here, we shall give another proof.

Theorem 4.6. *For a paracompact space X, $A_2(X)$ is a k-space if and only if X^2 is a k-space.*

Proof. First, we show the "only if" part. Since X^2 is a clopen subset of $(X \oplus -X \oplus \{0\})^2$ and $X^2 \subset i_2^{-1}(A_2(X) \setminus A_1(X))$, by Lemma 4.5, $i_2(X^2)$ is a clopen subset of more,

$A_2(X)\setminus A_1(X)$. In particular, $i_2(X^2)$ is open in $A_2(X)$. Since $A_2(X)$ is a k-space, $i_2(X^2)$ is a k-space. On the other hand, since $i_2^{-1}(i_2(X^2)) = X^2$, by Lemma 4.5, the mapping $i_2|_{X^2}: X^2 \to i_2(X^2)$ is a perfect mapping. Thus, X^2 is a k-space.

Next, we show the "if" part. Let E be a subset of $A_2(X)$ such that $E \cap K$ is closed in K for each compact subset K of $A_2(X)$, and take a word $g \in \bar{E}$. We shall show that $g \in E$. The proof is in three cases.

Case 1: $g \in A_2(X)\setminus A_1(X)$.

Since $A_2(X)\setminus A_1(X)$ is open in $A_2(X)$, there is an open neighborhood U of g in $A_2(X)$ such that $\bar{U} \subset A_2(X)\setminus A_1(X)$. We put $H = E \cap \bar{U}$ then we have that $g \in \bar{H}$ and $H \cap K$ is closed in K for each compact subset of $\bar{U}$. On the other hand, by Lemma 4.5, $i_2|_{i_2^{-1}(\bar{U})}: i_2^{-1}(\bar{U}) \to \bar{U}$ is an open mapping, and $i_2^{-1}(\bar{U})$ is a closed subset of the k-space X^2. It follows that $\bar{U}$ is a k-space. Therefore, H is closed in $\bar{U}$, and hence in $A_2(X)$. Consequently, $g \in H \subset E$.

Case 2: $g \in X \oplus -X$.

Recall that A_0 is a clopen subgroup of $A(X)$, then $g + A_0$ is a clopen neighborhood of g in $A(X)$. Note that $(g + A_0) \cap ((A_2(X)\setminus A_1(X)) \cup \{0\}) = \emptyset$. We put $H = (g + A_0) \cap E$, then it is easy to see that $g \in \bar{H} \subset X \oplus -X$ and $H \cap K$ is closed in K for each compact subset K of $X \oplus -X$. Since $X \oplus -X$ is a k-space, H is closed in $X \oplus -X$. Hence $g \in H \subset E$.

Case 3: $g = 0$.

Assume that $0 \in \bar{E}\setminus E$. Set $H = E \cap A_0$, then we can see that $0 \in \overline{H}\setminus H$ and $H \subset A_2(X)\setminus A_1(X)$. Moreover $\overline{j_1^{-1}(H)} \subset X^2\setminus \Delta_X$ and by Corollary 2.6, $\overline{j_1^{-1}(H)} \cap \Delta_X \neq \emptyset$. Take a point $\boldsymbol{x} = (x, x) \in \overline{j_1^{-1}(H)} \cap \Delta_X$. Let C be an arbitrary compact subset of X^2. Then $j_1(C)$ is a compact subset of $A_2(X)$. By the assumption, $H \cap j_1(C)$ is closed in $j_1(C)$, so in $A_2(X)$. Thus, $j_1^{-1}(H \cap j_1(C))$ is closed in X^2. Since $j_1|_{j_1^{-1}((A_2(X)\setminus A_1(X))\cap A_0)}: X^2\setminus\Delta_X = j_1^{-1}((A_2(X)\setminus A_1(X)) \cap A_0) \to (A_2(X)\setminus A_1(X)) \cap A_0$ is one-to-one and onto, and $H \cap j_1(C) \subset (A_2(X)\setminus A_1(X)) \cap A_0$, we can see that $j_1^{-1}(H \cap j_1(C)) = j_1^{-1}(H) \cap C$. Thus $j_1^{-1}(H) \cap C$ is closed in X^2, so in C. It follows that $j_1^{-1}(H)$ is closed in X^2 because X^2 is a k-space. This contradicts that $\boldsymbol{x} \in \overline{j_1^{-1}(H)}\setminus j_1^{-1}(H)$. Consequently, we have $0 \in E$. □

Remark 3.7 and Theorem 4.6 yield

Corollary 4.7. *$A_2(V(\aleph_1))$ is not a k-space.*

Corollary 4.8. *For a metrizable space X, $A_2(X)$ is a k-space, and the mapping i_2 is quotient.*

Theorem 4.9. *If X is a metrizable space, then the following statements are equivalent:*

(a) *$A_3(X)$ is a k-space,*

(b) *the mapping i_3 is quotient,*

(c) *X is locally compact or the set X' of all nonisolated points in X is compact.*

Proof. By Proposition 4.1, (a) and (b) are equivalent. In the proof of the implication (b)⇒(e) in Theorem 4.2, it was already shown the implication (a)⇒(c). Furthermore, by the proof of the implication (e)$\Rightarrow$(a) in Theorem 4.2, $A_3(X)$ is a k-space if X' is compact. Thus, to complete the proof, it suffices to show that $A_3(X)$ is a k-space if X is locally compact.

Let X be a locally compact metrizable space. Then X can be represented as the sum of locally compact separable spaces, i.e., $X=\bigoplus\{X_\alpha : \alpha\in A\}$, where each X_α is locally compact separable. For each $\alpha\in A$, let $\mathcal{U}_\alpha$ be the universal uniformity on X_α and Δ_α the diagonal of $X_\alpha\times X_\alpha$. Then, $\mathcal{U}=\{\bigoplus_{\alpha\in A} U_\alpha : \in \mathcal{U}_\alpha\}$ is the universal uniformity on X. To prove that $A_3(X)$ is a k-space, take a subset E of $A_3(X)$ such that $E\cap K$ is closed in K for each compact subset K of $A_3(X)$, and an arbitrary point $g\in\bar{E}$. We shall show that $g\in E$. If $g\in(A_3(X)\backslash A_2(X))\cup(A_2(X)\backslash A_1(X))\cup\{0\}$, it can be shown that $g\in E$ in the similar way to the proof of Theorem 4.6. Thus we may assume that $g\in X\oplus -X$, in particular, $g\in X$ (if $g\in -X$, we can show similarly).

We put $H=(E-g)\cap A_0$, then H is a subset of $A_4(X)$ such that $H\cap K$ is closed in K for each compact subset K of $A_4(X)$ and $0\in\bar{H}$. If $0\in\overline{H\cap A_2(X)}$, then it can be shown that $0\in H\cap A_2(X)$ and $g\in H$ from the proof of Theorem 4.6. Thus, we may assume that $H\subset(A_4(X)\backslash A_3(X))\cup\{0\}$ because $H\cap(A_3(X)\backslash A_2(X))=\emptyset$. By the definition of H, we put

$$H=\{h_\lambda=x_\lambda-y_\lambda+z_\lambda-g : \lambda\in\Lambda\},$$

where $x_\lambda, y_\lambda, z_\lambda\in X$ for each $\lambda\in\Lambda$. Since $g\in X$, there is an $\alpha_0\in A$ such that $g\in X_{\alpha_0}$. Let

$$H_1=\{h_\lambda\in H : x_\lambda\in X_{\alpha_0} \text{ or } z_\lambda\in X_{\alpha_0}\}.$$

Hence it is easy to see that $0\notin\overline{H\backslash H_1}$ because $j_2^{-1}(H\backslash H_1)\cap U^2=\emptyset$ for each $U\in\mathcal{U}$. Thus we have $0\in\overline{H_1}$. Now, we assume that $0\in\overline{H_2}$, where

$$H_2=\{h_\lambda\in H_1 : x_\lambda, z_\lambda\in X_{\alpha_0}\}.$$

Then $j_2^{-1}(H_2)\cap U^2\neq\emptyset$ for each $U\in\mathcal{U}$. By the definition of $\mathcal{U}$, $j_2^{-1}(H_3)\cap U^2\neq\emptyset$ for each $U\in\mathcal{U}$, where

$$H_3=\{h_\lambda\in H_2 : y_\lambda\in X_{\alpha_0}\}.$$

It follows that $0\in\overline{H_3}$. On the other hand, H_3 is a subset of $A_4(X_{\alpha_0})$ and $A_4(X_{\alpha_0})$ can be considered as a closed subset of $A_4(X)$ by Lemma 1.1(4). Since X_{α_0} is locally compact separable, by Theorem 4.2, $A_4(X_{\alpha_0})$ is a k-space. Furthermore, $H_3\subset H\cap A_4(X_{\alpha_0})$ and $H\cap A_4(X_{\alpha_0})\cap K$ is closed in K for each compact subset K of $A_4(X_{\alpha_0})$. We can easily see that $0\in H\cap A_4(X_{\alpha_0})\subset H$. Therefore it suffices to show that $0\in H$ if $0\in\overline{H_1\backslash H_2}$. Let

$$H_4=\{h_\lambda : x_\lambda\notin X_{\alpha_0} \text{ and } z_\lambda\in X_{\alpha_0}\},$$

and we may assume that $0\in\overline{H_4}$. Now, we put

$$\Lambda'=\{\lambda\in\Lambda : h_\lambda\in H_4\},$$

$$L=\{(z_\lambda, g) : \lambda\in\Lambda'\},$$

and

$$M=\{(x_\lambda, y_\lambda) : \lambda\in\Lambda'\}.$$

Since $j_2^{-1}(H_4) \cap U^2 \neq \emptyset$ for each $u \in \mathcal{U}$,

(1) $L \cap U \neq \emptyset$ for each $U \in \mathcal{U}$, and

(2) $M \cap U \neq \emptyset$ for each $U \in \mathcal{U}$.

Since $L \subset X^2_{\alpha_0}$, by (1), $\bar{L} \cap \Delta_{\alpha_0} \neq \emptyset$ and, in particular, we can see that $(g, g) \in \bar{L}$. Therefore $g \in \overline{\{z_\lambda : \lambda \in \Lambda'\}}$. Now, let $\mathcal{B}_g = \{B_n : n \in \mathbb{N}\}$ be a countable neighborhood base of g in X_{α_0} such that $B_{n+1} \subset B_n$ and $B_1 = X_{\alpha_0}$. For each $n \in \mathbb{N}$, let

$$M_n = \{(x_\lambda, y_\lambda) : z_\lambda \in B_n \setminus B_{n+1}\}.$$

Then $M = \bigcup_{n=1}^\infty M_n$. On the other hand, by (2), $\bar{M} \cap \Delta_X \neq \emptyset$. We consider the following two cases.

Case 1: *There is a subsequence* $\{k_m : m \in \mathbb{N}\}$ *of* $\mathbb{N}$ *such that* $\overline{M_{k_m}} \cap \Delta_X \neq \emptyset$ *for each* $m \in \mathbb{N}$.

For each $m \in \mathbb{N}$, we can take an $\alpha_m \in A$ such that $\overline{M_{k_m}} \cap \Delta_{\alpha_m} \neq \emptyset$. We put $N_{\alpha_m} = M_{k_m} \cap X^2_{\alpha_m}$, then $\overline{N_{\alpha_m}} \cap \Delta_{\alpha_m} \neq \emptyset$. Thus, $j_2^{-1}(H_5) \cap \Delta_Y \neq \emptyset$, so that $0 \in \overline{H_5}$, where

$$H_5 = \left\{ h_\lambda \in H_4 : z_\lambda \in B_{k_1} \text{ and } (x_\lambda, y_\lambda) \in \bigcup_{m=1}^{\infty} N_{\alpha_m} \right\}$$

and

$$Y = \bigoplus_{m=0}^{\infty} X_{\alpha_m}.$$

Since Y is a locally compact separable metrizable closed subspace of X, $A_4(Y)$ is closed in $A_4(X)$ and, by Theorem 4.2, $A_4(Y)$ is a k-space. Furthermore, $H_5 \subset H \cap A_4(Y)$ and $H \cap A_4(Y) \cap K$ is closed in K for each compact subset K of $A_4(Y)$. It follows that $0 \in H \cap A_4(Y) \subset H$.

Case 2: *There is an* $n \in \mathbb{N}$ *such that* $\overline{M_m} \cap \Delta_X = \emptyset$ *for each* $m \geqslant n$.

Let

$$H_6 = \{h_\lambda : z_\lambda \in B_n\}$$

and

$$M(n) = \bigcup_{m \geqslant n} M_m.$$

Then it is easy to see that $0 \in \overline{H_6}$, so that $\overline{M(n)} \cap \Delta_X \neq \emptyset$. Thus, we can take an $\alpha \in A$ such that $\overline{M(n)} \cap \Delta_\alpha \neq \emptyset$. Now, we can assume that $N_m = M_m \cap X_\alpha \neq \emptyset$ for each $m \geqslant n$. Hence, it can be seen that for each $U \in \mathcal{U}_\alpha$, $\{m \geqslant n : N_m \cap U \neq \emptyset\}$ is an infinite set because $\overline{N_m} \cap \Delta_\alpha = \emptyset$ for each $m \geqslant n$ and $\overline{\bigcup_{m \geqslant n} N_m} \cap \Delta_\alpha \neq \emptyset$. It follows that $0 \in \overline{H_7}$, where

$$H_7 = \left\{ h_\lambda \in H_6 : (x_\lambda, y_\lambda) \in \bigcup_{m \geqslant n} N_m \right\}.$$

On the other hand, $H_7 \subset A_4(X_{\alpha_0} \oplus X_\alpha)$, $A_4(X_{\alpha_0} \oplus X_\alpha)$ is closed in $A_4(X)$ and $A_4(X_{\alpha_0} \oplus X_\alpha)$ is a k-space. Therefore, in the same argument as in Case 1, we can see that $0 \in H \cap A_4(X_{\alpha_0} \oplus X_\alpha) \subset H$.

Consequently, in any case, we can prove that $0 \in H$, so that $g \in E$. Thus, E is a closed subset of $A_3(X)$. It follows that $A_3(X)$ is a k-space. $\square$

Corollary 4.10. *Let M_3 be the space constructed in Section 3, then $A_3(M_3)$ is a k-space but $A_4(M_3)$ is not a k-space.*

5. Applications

Finally, we discuss about the questions which are asked by Fay, Ordman and Thomas in [5]. Since they discussed about the free topological group $F_G(X)$ over a space in the sense of Graev, we introduce some notations. For each $n \in \mathbb{N}$, we denote the subspace of $F_G(X)$ consisting of words of length not exceeding n by $F_G(X)_n$. Since the unit element e of $F_G(X)$ is a point of X, the mapping i_n is defined on $X \cup X^{-1}$. Analogously, let $A_G(X)$ be the free Abelian topological group over a space X in the sense of Graev, and the subspace $A_G(X)_n$ and the mapping i_n can be defined similarly to these for $F_G(X)$. Fay, Ordman and Thomas [5] asked the following questions.

Question 5.1. Is i_n always a quotient map if X is locally compact?

Question 5.2. Is $i_2 : (Q \cup Q^{-1})^2 \to F_G(Q)_2$ a quotient map? Is i_2 always a quotient map?

Pestov [13] answered these questions as follows.

Theorem 5.3. (1) *The mapping $i_2 : (X \oplus X^{-1} \oplus \{e\})^n \to F_n(X)$ $((X \cup X^{-1})^n \to F_G(X)_n)$ is quotient in either sense if and only if each neighborhood of Δ_X in X^2 is in $\mathcal{U}_X$. In particular, if X is paracompact, i_2 is quotient. Hence, the first part of Question* 5.2 *is yes.*

(2) *Let X be a locally compact space which is not paracompact, then i_2 is not quotient by* (1). *Hence, Question* 5.1 *is answered by no.*

On the other hand, about the space M_3 described in Section 3, we note the following fact, that implies the negative answer of the Abelian version of Question 5.1 even if a space X is locally compact and metrizable.

Theorem 5.4. *The space M_3 is a locally compact metrizable space such that $A_4(M_3)$ is not a k-space and $A_n(M_3)$ is homeomorphic to $A_G(M_3)_n$ for each $n \in \mathbb{N}$. Thus, the mapping i_n is not quotient in either sense.*

Proof. It suffices to show that $A_n(M_3)$ is homeomorphic to $A_G(M_3)_n$ for each $n \in \mathbb{N}$. Since M_3 has infinite many isolated points, for each isolated point x in X, M_3 is homeomorphic to $M_3 \backslash \{x\} = M_3'$. On the other hand, from the argument in [6], $A_G(M_3)$ is topologically isomorphic to $A(M_3')$, respectively. In fact, the topological isomorphism is the continuous homomorphic extension of the identity on M_3, so that we can prove that $A_G(M_3)_n$ is homeomorphic to $A_n(M_3')$ for each $n \in \mathbb{N}$. Thus, it follows that $A_n(M_3)$ is homeomorphic to $A_G(M_3)_n$ for each $n \in \mathbb{N}$. $\square$

Acknowledgement

The author wishes to thank Professor A. Koyama and the referee for carefully reading preliminary versions of this paper. They provided many valuable suggestions.

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